

THE RANGE OF TREE-INDEXED RANDOM WALK IN LOW DIMENSIONS

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We study the range R_n of a random walk on the d -dimensional lattice \mathbb{Z}^d indexed by a random tree with n vertices. Under the assumption that the random walk is centered and has finite fourth moments, we prove in dimension $d \leq 3$ that $n^{-d/4}R_n$ converges in distribution to the Lebesgue measure of the support of the integrated super-Brownian excursion (ISE). An auxiliary result shows that the suitably rescaled local times of the tree-indexed random walk converge in distribution to the density process of ISE. We obtain similar results for the range of critical branching random walk in \mathbb{Z}^d , $d \leq 3$. As an intermediate estimate, we get exact asymptotics for the probability that a critical branching random walk starting with a single particle at the origin hits a distant point. The results of the present article complement those derived in higher dimensions in our earlier work.

1. Introduction. In the present paper, we continue our study of asymptotics for the number of distinct sites of the lattice visited by a tree-indexed random walk. We consider (discrete) plane trees, which are rooted ordered trees that can be viewed as describing the genealogy of a population starting with one ancestor or root, which is denoted by the symbol \emptyset . Given such a tree \mathcal{T} and a probability measure θ on \mathbb{Z}^d , we can consider the random walk with jump distribution θ indexed by the tree \mathcal{T} . This means that we assign a (random) spatial location $Z_{\mathcal{T}}(u) \in \mathbb{Z}^d$ to every vertex u of \mathcal{T} , in the following way. First, the spatial location $Z_{\mathcal{T}}(\emptyset)$ of the root is the origin of \mathbb{Z}^d . Then we assign independently to every edge e of the tree \mathcal{T} a random variable Y_e distributed according to θ , and we let the spatial location $Z_{\mathcal{T}}(u)$ of the vertex u be the sum of the quantities Y_e over all edges e belonging to the simple path from \emptyset to u in the tree. The number of distinct spatial locations is called the range of the tree-indexed random walk $Z_{\mathcal{T}}$.

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In our previous work [13], we stated the following result. Let θ be a probability distribution on \mathbb{Z}^d , which is symmetric with finite support and is not supported on a strict subgroup of \mathbb{Z}^d , and for every integer $n \geq 1$, let \mathcal{T}_n° be a random tree uniformly distributed over all plane trees with n vertices. Conditionally given \mathcal{T}_n° , let $Z_{\mathcal{T}_n^\circ}$ be a random walk with jump distribution θ indexed by \mathcal{T}_n° , and let R_n stand for the range of $Z_{\mathcal{T}_n^\circ}$. Then:

- if $d \geq 5$,

$$\frac{1}{n} R_n \xrightarrow[n \rightarrow \infty]{(P)} c_\theta,$$

where $c_\theta > 0$ is a constant depending on θ , and $\xrightarrow{(P)}$ indicates convergence in probability;

- if $d = 4$,

$$\frac{\log n}{n} R_n \xrightarrow[n \rightarrow \infty]{L^2} 8\pi^2 \sigma^4,$$

where $\sigma^2 = (\det M_\theta)^{1/4}$, with M_θ denoting the covariance matrix of θ ;

- if $d \leq 3$,

$$(1) \quad n^{-d/4} R_n \xrightarrow[n \rightarrow \infty]{(d)} c_\theta \lambda_d(\text{supp}(\mathcal{I})),$$

where $c_\theta = 2^{d/4} (\det M_\theta)^{1/2}$ is a constant depending on θ , and $\lambda_d(\text{supp}(\mathcal{I}))$ stands for the Lebesgue measure of the support of the random measure on \mathbb{R}^d known as Integrated Super-Brownian Excursion or ISE (see Section 2.3 below for a definition of ISE in terms of the Brownian snake, and note that our normalization is slightly different from the one in [1]).

Only the cases $d \geq 5$ and $d = 4$ were proved in [13], in fact in a greater generality than stated above, especially when $d \geq 5$. In the present work, we concentrate on the case $d \leq 3$ and we prove a general version of the convergence (1), where instead of considering a uniformly distributed plane tree with n vertices we deal with a Galton–Watson tree with offspring distribution μ conditioned to have n vertices.

Let us specify the assumptions that will be in force throughout this article. We always assume that $d \leq 3$ and:

- μ is a nondegenerate critical offspring distribution on \mathbb{Z}_+ , such that, for some $\lambda > 0$,

$$\sum_{k=0}^{\infty} e^{\lambda k} \mu(k) < \infty,$$

and we set $\rho := (\text{var } \mu)^{1/2} > 0$;

- θ is a probability measure on \mathbb{Z}^d , which is not supported on a strict subgroup of \mathbb{Z}^d ; θ is such that

$$(2) \quad \lim_{r \rightarrow +\infty} r^4 \theta(\{x \in \mathbb{Z}^d : |x| > r\}) = 0,$$

and θ has zero mean; we set $\sigma := (\det M_\theta)^{1/2d} > 0$, where M_θ denotes the covariance matrix of θ .

Note that (2) holds if θ has finite fourth moments.

For every $n \geq 1$ such that this makes sense, let \mathcal{T}_n be a Galton–Watson tree with offspring distribution μ conditioned to have n vertices. Note that the case when \mathcal{T}_n is uniformly distributed over plane trees with n vertices is recovered when μ is the geometric distribution with parameter $1/2$ (see, e.g., Section 2.2 in [14]). Let $Z_{\mathcal{T}_n}$ denote the random walk with jump distribution θ indexed by \mathcal{T}_n , and let R_n be the range of $Z_{\mathcal{T}_n}$. Theorem 5 below shows that the convergence (1) holds, provided that c_θ is replaced by the constant $2^{d/2} \sigma^d \rho^{-d/2}$.

An interesting auxiliary result is an invariance principle for “local times” of our tree-indexed random walk. For every $a \in \mathbb{Z}^d$, let

$$L_n(a) = \sum_{u \in \mathcal{T}_n} \mathbf{1}_{\{Z_{\mathcal{T}_n}(u)=a\}}$$

be the number of visits of a by the tree-indexed random walk $Z_{\mathcal{T}_n}$. For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, set $\lfloor x \rfloor := (\lfloor x_1 \rfloor, \dots, \lfloor x_d \rfloor)$. Then Theorem 4 shows that the process

$$(n^{d/4-1} L_n(\lfloor n^{1/4} x \rfloor))_{x \in \mathbb{R}^d \setminus \{0\}}$$

converges as $n \rightarrow \infty$, in the sense of weak convergence of finite-dimensional marginals, to the density process of ISE (up to scaling constants and a linear transformation of the variable x). Notice that the latter density process exists because $d \leq 3$, by results due to Sugitani [19]. In dimension $d = 1$, this invariance principle has been obtained earlier in a stronger (functional) form by Bousquet-Mélou and Janson [3], Theorem 3.6, in a particular case, and then by Devroye and Janson [5], Theorem 1.1, in a more general setting. Such a strengthening might also be possible when $d = 2$ or 3 , but we have chosen not to investigate this question here as it is not relevant to our main applications. In dimensions 2 and 3, Lalley and Zheng [7], Theorem 1, also give a closely related result for local times of critical branching random walk in the case of a Poisson offspring distribution and for a particular choice of θ .

Our tree-indexed random walk can be viewed as a branching random walk starting with a single initial particle and conditioned to have a fixed total progeny. Therefore, it is not surprising that our main results have analogs for branching random walks, as it was already the case in dimension $d \geq 4$ (see

Propositions 20 and 21 in [13]). For every integer $p \geq 1$, consider a (discrete time) branching random walk starting initially with p particles located at the origin of \mathbb{Z}^d , such that the offspring number of each particle is distributed according to μ , and each newly born particle jumps from the location of its parent according to the jump distribution θ . Let $\mathcal{V}^{[p]}$ stand for the set of all sites of \mathbb{Z}^d visited by this branching random walk. Then Theorem 8 shows that, similarly as in (1), the asymptotic distribution of $p^{-d/2} \# \mathcal{V}^{[p]}$ is the Lebesgue measure of the range of a super-Brownian motion starting from δ_0 (note again that this Lebesgue measure is positive because $d \leq 3$, see [4] or [19]). In a related direction, we mention the article of Lalley and Zheng [8], which gives estimates for the number of occupied sites *at a given time* by a critical nearest neighbor branching random walk in \mathbb{Z}^d .

Our proof of Theorem 8 depends on an asymptotic estimate for the hitting probability of a distant point by branching random walk, which seems to be new and of independent interest. To be specific, consider the set $\mathcal{V}^{[1]}$ of all sites visited by the branching random walk starting with a single particle at the origin. Consider for simplicity the isotropic case where $M_\theta = \sigma^2 \text{Id}$, where Id is the identity matrix. Then Theorem 7 shows that

$$\lim_{|a| \rightarrow \infty} |a|^2 P(a \in \mathcal{V}^{[1]}) = \frac{2(4-d)\sigma^2}{\rho^2}.$$

See Section 5.1 for a discussion of similar estimates in higher dimensions.

Not surprisingly, our proofs depend on the known relations between tree-indexed random walk (or branching random walk) and the Brownian snake (or super-Brownian motion). In particular, we make extensive use of a result of Janson and Marckert [6] showing that the “discrete snake” coding our tree-indexed random walk $Z_{\mathcal{T}_n}$ converges in distribution in a strong (functional) sense to the Brownian snake driven by a normalized Brownian excursion. It follows from this convergence that the set of all sites visited by the tree-indexed random walk converges in distribution (modulo a suitable rescaling) to the support of ISE, in the sense of the Hausdorff distance between compact sets. But, of course, this is not sufficient to derive asymptotics for the *number* of visited sites.

Our assumptions on μ and θ are similar to those in [6]. We have not striven for the greatest generality, and it is plausible that these assumptions can be relaxed. See, in particular, [6] for a discussion of the necessity of the existence of exponential moments for the offspring distribution μ . It might also be possible to replace our condition (2) on θ by a second moment assumption, but this would require different methods as the results of [6] show that the strong convergence of discrete snakes to the Brownian snake no longer holds without (2).

The paper is organized as follows. Section 2 presents our main notation and gives some preliminary results about the Brownian snake. Section 3 is

devoted to our main result about the range of tree-indexed random walk in dimension $d \leq 3$. Section 4 discusses similar results for branching random walk, and Section 5 presents a few complements and open questions.

2. Preliminaries on trees and the Brownian snake.

2.1. *Finite trees.* We use the standard formalism for plane trees. We set

$$\mathcal{U} := \bigcup_{n=0}^{\infty} \mathbb{N}^n,$$

where $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}^0 = \{\emptyset\}$. If $u = (u_1, \dots, u_n) \in \mathcal{U}$, we set $|u| = n$ [in particular $|\emptyset| = 0$]. We write \prec for the lexicographical order on \mathcal{U} , so that $\emptyset \prec 1 \prec (1, 1) \prec 2$, for instance.

A plane tree (or rooted ordered tree) \mathcal{T} is a finite subset of \mathcal{U} such that:

- (i) $\emptyset \in \mathcal{T}$;
- (ii) If $u = (u_1, \dots, u_n) \in \mathcal{T} \setminus \{\emptyset\}$ then $\tilde{u} := (u_1, \dots, u_{n-1}) \in \mathcal{T}$;
- (iii) For every $u = (u_1, \dots, u_n) \in \mathcal{T}$, there exists an integer $k_u(\mathcal{T}) \geq 0$ such that, for every $j \in \mathbb{N}$, $(u_1, \dots, u_n, j) \in \mathcal{T}$ if and only if $1 \leq j \leq k_u(\mathcal{T})$.

The notions of a descendant or of an ancestor of a vertex of \mathcal{T} are defined in an obvious way. If $u, v \in \mathcal{T}$, we will write $u \wedge v \in \mathcal{T}$ for the most recent common ancestor of u and v . We denote the set of all planes trees by \mathbb{T}_f .

Let \mathcal{T} be a tree with $p = \#\mathcal{T}$ vertices and let $\emptyset = v_0 \prec v_1 \prec \dots \prec v_{p-1}$ be the vertices of \mathcal{T} listed in lexicographical order. We define the height function $(H_i)_{0 \leq i \leq p}$ of \mathcal{T} by setting $H_i = |v_i|$ for every $0 \leq i \leq p-1$, and $H_p = 0$ by convention.

Recall that we have fixed a probability measure μ on \mathbb{Z}_+ satisfying the assumptions given in Section 1, and that $\rho^2 = \text{var } \mu$. The law of the Galton–Watson tree with offspring distribution μ is a probability measure on the space \mathbb{T}_f , which is denoted by Π_μ (see, e.g., [12], Section 1).

We will need some information about the law of the total progeny $\#\mathcal{T}$ under Π_μ . It is well known (see, e.g., [12], Corollary 1.6) that this law is the same as the law of the first hitting time of -1 by a random walk on \mathbb{Z} with jump distribution $\nu(k) = \mu(k+1)$, $k = -1, 0, 1, \dots$ started from 0. Combining this with Kemperman’s formula (see, e.g., [17], page 122) and using a standard local limit theorem, one gets

$$(3) \quad \lim_{k \rightarrow \infty} k^{1/2} \Pi_\mu(\#\mathcal{T} \geq k) = \frac{2}{\rho\sqrt{2\pi}}.$$

Suppose that μ is not supported on a strict subgroup of \mathbb{Z} , so that the random walk with jump distribution ν is aperiodic. The preceding asymptotics can then be strengthened in the form

$$(4) \quad \lim_{k \rightarrow \infty} k^{3/2} \Pi_\mu(\#\mathcal{T} = k) = \frac{1}{\rho\sqrt{2\pi}}.$$

2.2. Tree-indexed random walk. A (d -dimensional) spatial tree is a pair $(\mathcal{T}, (z_u)_{u \in \mathcal{T}})$ where $\mathcal{T} \in \mathbb{T}_f$ and $z_u \in \mathbb{Z}^d$ for every $u \in \mathcal{T}$. We let \mathbb{T}_f^* be the set of all spatial trees.

Recall that θ is a probability measure on \mathbb{Z}^d satisfying the assumptions listed in the [Introduction](#). We write $\Pi_{\mu, \theta}^*$ for the probability distribution on \mathbb{T}_f^* under which \mathcal{T} is distributed according to Π_μ and, conditionally on \mathcal{T} , the “spatial locations” $(z_u)_{u \in \mathcal{T}}$ are distributed as random walk indexed by \mathcal{T} , with jump distribution θ , and started from 0 at the root \emptyset : This means that, under the probability measure $\Pi_{\mu, \theta}^*$, we have $z_\emptyset = 0$ a.s. and, conditionally on \mathcal{T} , the quantities $(z_u - z_{\tilde{u}}, u \in \mathcal{T} \setminus \{\emptyset\})$ are independent and distributed according to θ .

2.3. The Brownian snake. We refer to [\[11\]](#) for the basic facts about the Brownian snake that we will use. The Brownian snake $(W_s)_{s \geq 0}$ is a Markov process taking values in the space \mathcal{W} of all (d -dimensional) stopped paths: Here, a stopped path w is just a continuous mapping $w: [0, \zeta_{(w)}] \rightarrow \mathbb{R}^d$, where the number $\zeta_{(w)} \geq 0$, which depends on w , is called the lifetime of w . A stopped path w with zero lifetime will be identified with its starting point $w(0) \in \mathbb{R}^d$. The endpoint $w(\zeta_{(w)})$ of a stopped path w is denoted by \hat{w} .

It will be convenient to argue on the canonical space $C(\mathbb{R}_+, \mathcal{W})$ of all continuous mappings from \mathbb{R}_+ into \mathcal{W} , and to let $(W_s)_{s \geq 0}$ be the canonical process on this space. We write $\zeta_s := \zeta_{(W_s)}$ for the lifetime of W_s . If $x \in \mathbb{R}^d$, the law of the Brownian snake starting from x is the probability measure \mathbb{P}_x on $C(\mathbb{R}_+, \mathcal{W})$ that is characterized as follows:

- (i) The distribution of $(\zeta_s)_{s \geq 0}$ under \mathbb{P}_x is the law of a reflected linear Brownian motion on \mathbb{R}_+ started from 0.
- (ii) We have $W_0 = x$, \mathbb{P}_x a.s. Furthermore, under \mathbb{P}_x and conditionally on $(\zeta_s)_{s \geq 0}$, the process $(W_s)_{s \geq 0}$ is (time-inhomogeneous) Markov with transition kernels specified as follows. If $0 \leq s < s'$,
 - $W_{s'}(t) = W_s(t)$ for every $0 \leq t \leq m_\zeta(s, s') := \min\{\zeta_r : s \leq r \leq s'\}$;
 - $(W_{s'}(m_\zeta(s, s') + t) - W_{s'}(m_\zeta(s, s')))_{0 \leq t \leq \zeta_{s'} - m_\zeta(s, s')}$ is a standard Brownian motion in \mathbb{R}^d independent of W_s .

We will refer to the process $(W_s)_{s \geq 0}$ under \mathbb{P}_0 as the standard Brownian snake.

We will also be interested in (infinite) excursion measures of the Brownian snake, which we denote by \mathbb{N}_x , $x \in \mathbb{R}^d$. For every $x \in \mathbb{R}^d$, the distribution of the process $(W_s)_{s \geq 0}$ under \mathbb{N}_x is characterized by properties analogous to (i) and (ii) above, with the only difference that in (i) the law of reflected linear Brownian motion is replaced by the Itô measure of positive excursions of linear Brownian motion, normalized in such a way that $\mathbb{N}_x(\sup\{\zeta_s : s \geq 0\} > \varepsilon) = (2\varepsilon)^{-1}$, for every $\varepsilon > 0$.

We write $\gamma := \sup\{s \geq 0 : \zeta_s > 0\}$, which corresponds to the duration of the excursion under \mathbb{N}_x . A special role will be played by the probability measures $\mathbb{N}_x^{(r)} := \mathbb{N}_x(\cdot | \gamma = r)$, which are defined for every $x \in \mathbb{R}^d$ and every $r > 0$. Under $\mathbb{N}_x^{(r)}$, the “lifetime process” $(\zeta_s)_{0 \leq s \leq r}$ is a Brownian excursion with duration r . From the analogous decomposition for the Itô measure of Brownian excursions, we have

$$(5) \quad \mathbb{N}_0 = \int_0^\infty \frac{dr}{2\sqrt{2\pi r^3}} \mathbb{N}_0^{(r)}.$$

The total occupation measure of the Brownian snake is the finite measure \mathcal{Z} on \mathbb{R}^d defined under \mathbb{N}_x , or under $\mathbb{N}_x^{(r)}$, by the formula

$$\langle \mathcal{Z}, \varphi \rangle = \int_0^\gamma ds \varphi(\widehat{W}_s),$$

for any nonnegative measurable function φ on \mathbb{R}^d .

Under $\mathbb{N}_x^{(1)}$, \mathcal{Z} is a random probability measure, which in the case $x = 0$ is called ISE for integrated super-Brownian excursion [the measure \mathcal{I} in (1)] is thus distributed as \mathcal{Z} under $\mathbb{N}_0^{(1)}$. Note that our normalization of ISE is slightly different from the one originally proposed by Aldous [1].

The following result will be derived from known properties of super-Brownian motion via the connection between the Brownian snake and superprocesses.

PROPOSITION 1. *Both \mathbb{N}_x a.e. and $\mathbb{N}_x^{(1)}$ a.s., the random measure \mathcal{Z} has a continuous density on \mathbb{R}^d , which will be denoted by $(\ell^y, y \in \mathbb{R}^d)$.*

REMARK. When $d = 1$, this result, under the measure $\mathbb{N}_0^{(1)}$, can be found in [3], Theorem 2.1.

PROOF OF PROPOSITION 1. By translation invariance, it is enough to consider the case $x = 0$. We rely on the Brownian snake construction of super-Brownian motion to deduce the statement of the proposition from Sugitani’s results [19]. Let $(W^i)_{i \in I}$ be a Poisson point measure on $C(\mathbb{R}_+, \mathcal{W})$ with intensity \mathbb{N}_0 . With every $i \in I$, we associate the occupation measure \mathcal{Z}^i of W^i . Then Theorem IV.4 in [11] shows that there exists a super-Brownian motion $(X_t)_{t \geq 0}$ with branching mechanism $\psi(u) = 2u^2$ and initial value $X_0 = \delta_0$, such that

$$\int_0^\infty dt X_t = \sum_{i \in I} \mathcal{Z}^i.$$

As a consequence of [19], Theorems 2 and 3, the random measure $\int_0^\infty dt X_t$ has a.s. a continuous density on $\mathbb{R}^d \setminus \{0\}$. On the other hand, let $B(0, \varepsilon)$

denote the closed ball of radius ε centered at 0 in \mathbb{R}^d . Then, for every $\varepsilon > 0$, the event

$$\mathcal{A}_\varepsilon := \{\#\{i \in I : \mathcal{Z}^i(B(0, \varepsilon)^c) > 0\} = 1\}$$

has positive probability (see, e.g., [11], Proposition V.9). On the event \mathcal{A}_ε , write i_0 for the unique index in I such that $\mathcal{Z}^{i_0}(B(0, \varepsilon)^c) > 0$. Then, still on the event \mathcal{A}_ε , the measures $\int_0^\infty dt X_t$ and \mathcal{Z}^{i_0} coincide on $B(0, \varepsilon)^c$. The conditional distribution of W^{i_0} knowing \mathcal{A}_ε is $\mathbb{N}_0(\cdot | \mathcal{Z}(B(0, \varepsilon)^c) > 0)$, and we conclude that \mathcal{Z} has a continuous density on $B(0, \varepsilon)^c$, $\mathbb{N}_0(\cdot | \mathcal{Z}(B(0, \varepsilon)^c) > 0)$ a.s. As this holds for any $\varepsilon > 0$, we obtain that, \mathbb{N}_0 a.e., the random measure \mathcal{Z} has a continuous density on $\mathbb{R}^d \setminus \{0\}$. Via a scaling argument, the same property holds $\mathbb{N}_0^{(1)}$ a.s. This argument does not exclude the possibility that \mathcal{Z} might have a singularity at 0, but we can use the rerooting invariance property (see [1], Section 3.2 or [15], Section 2.3) to complete the proof. According to this property, if under the measure $\mathbb{N}_0^{(1)}$ we pick a random point distributed according to \mathcal{Z} and then shift \mathcal{Z} so that this random point becomes the origin of \mathbb{R}^d , the resulting random measure has the same distribution as \mathcal{Z} . Consequently, we obtain that $\mathbb{N}_0^{(1)}$ a.s., $\mathcal{Z}(dx)$ a.e., the measure \mathcal{Z} has a continuous density on $\mathbb{R}^d \setminus \{x\}$. It easily follows that \mathcal{Z} has a continuous density on \mathbb{R}^d , $\mathbb{N}_0^{(1)}$ a.s., and by scaling again the same property holds under \mathbb{N}_0 . \square

Let us introduce the random closed set

$$\mathcal{R} := \{\widehat{W}_s : 0 \leq s \leq \gamma\}.$$

Note that, by construction, \mathcal{Z} is supported on \mathcal{R} , and it follows that, for every $y \in \mathbb{R}^d \setminus \{x\}$,

$$(6) \quad \{\ell^y > 0\} \subset \{y \in \mathcal{R}\}, \quad \mathbb{N}_x \text{ a.e. or } \mathbb{N}_x^{(1)} \text{ a.s.}$$

PROPOSITION 2. *For every $y \in \mathbb{R}^d \setminus \{x\}$,*

$$\{\ell^y > 0\} = \{y \in \mathcal{R}\}, \quad \mathbb{N}_x \text{ a.e. and } \mathbb{N}_x^{(1)} \text{ a.s.}$$

PROOF. Fix $y \in \mathbb{R}^d$, and consider the function $u(x) := \mathbb{N}_x(\ell^y > 0)$, for every $x \in \mathbb{R}^d \setminus \{y\}$. By simple scaling and rotational invariance arguments (see the proof of Proposition V.9(i) in [11] for a similar argument), we have

$$u(x) = C_d |x - y|^{-2}$$

with a certain constant $C_d > 0$ depending only on d . On the other hand, an easy application of the special Markov property [10] shows that, for every $r > 0$, and every $x \in B(y, r)^c$, we have

$$u(x) = \mathbb{N}_x \left[1 - \exp \left(- \int X^{B(y, r)^c}(dz) u(z) \right) \right],$$

where $X^{B(y,r)^c}$ stands for the exit measure of the Brownian snake from the open set $B(y,r)^c$. Theorem V.4 in [11] now shows that the function u must solve the partial differential equation $\Delta u = 4u^2$ in $\mathbb{R}^d \setminus \{y\}$. It easily follows that $C_d = 2 - d/2$.

The preceding line of reasoning also applies to the function $v(x) := \mathbb{N}_x(y \in \mathcal{R})$ (see [11], page 91), and shows that we have $v(x) = (2 - d/2)|x - y|^{-2} = u(x)$ for every $x \in \mathbb{R}^d \setminus \{y\}$ —note that this formula for v can also be derived from [4], Theorem 1.3 and the connection between the Brownian snake and super-Brownian motion. Recalling (6), this is enough to conclude that

$$(7) \quad \{\ell^y > 0\} = \{y \in \mathcal{R}\}, \quad \mathbb{N}_x \text{ a.e.}$$

for every $x \in \mathbb{R}^d \setminus \{y\}$.

We now want to obtain that the equality in (7) also holds $\mathbb{N}_x^{(1)}$ a.s. Note that, for every fixed x , we could use a scaling argument to get that $\{\ell^y > 0\} = \{y \in \mathcal{R}\}$, $\mathbb{N}_x^{(1)}$ a.s., for λ_d a.e. $y \in \mathbb{R}^d$, where we recall that λ_d stands for Lebesgue measure on \mathbb{R}^d . In order to get the more precise assertion of the proposition, we use a different method.

By translation invariance, we may assume that $x = 0$ and we fix $y \in \mathbb{R}^d \setminus \{0\}$. We set $T_y := \inf\{s \geq 0 : \widehat{W}_s = y\}$. Also, for every $s > 0$, we set

$$\tilde{\ell}_s^y := \liminf_{\varepsilon \rightarrow 0} (\lambda_d(B(y, \varepsilon)))^{-1} \int_0^s dr \mathbf{1}_{\{|\widehat{W}_r - y| \leq \varepsilon\}}.$$

Note that $\tilde{\ell}_s^y = \ell_s^y$, \mathbb{N}_0 a.e. and $\mathbb{N}_0^{(1)}$ a.e. We then claim that, for every $s > 0$,

$$(8) \quad \{T_y \leq s\} = \{\tilde{\ell}_s^y > 0\}, \quad \mathbb{N}_0 \text{ a.e.}$$

The inclusion $\{\tilde{\ell}_s^y > 0\} \subset \{T_y \leq s\}$ is obvious. In order to prove the reverse inclusion, we argue by contradiction and assume that

$$\mathbb{N}_0(T_y \leq s, \tilde{\ell}_s^y = 0) > 0.$$

Note that $\mathbb{N}_0(T_y = s) = 0$ [because $\mathbb{N}_0(\widehat{W}_s = y) = 0$], and so we have also $\mathbb{N}_0(T_y < s, \tilde{\ell}_s^y = 0) > 0$. For every $\eta > 0$, let

$$T_y^{(\eta)} := \inf\{r \geq T_y : \zeta_r \leq (\zeta_{T_y} - \eta)^+\}.$$

Notice that, by the properties of the Brownian snake, the path $W_{T_y^{(\eta)}}$ is just W_{T_y} stopped at time $(\zeta_{T_y} - \eta)^+$.

From the strong Markov property at time T_y , we easily get that $T_y^{(\eta)} \downarrow T_y$ as $\eta \downarrow 0$, \mathbb{N}_0 a.e. on $\{T_y < \infty\}$. Hence, on the event $\{T_y < s\}$, we have also $T_y^{(\eta)} < s$ for η small enough, \mathbb{N}_0 a.e. Therefore, we can find $\eta > 0$ such that

$$\mathbb{N}_0(T_y < s, \tilde{\ell}_{T_y^{(\eta)}}^y = 0) > 0.$$

However, using the strong Markov property at time $T_y^{(\eta)}$, and Lemma V.5 and Proposition V.9(i) in [11], we immediately see that, conditionally on the past up to time $T_y^{(\eta)}$, the event $\{\widehat{W}_r \neq y, \forall r \geq T_y^{(\eta)}\}$ occurs with positive probability. Hence, we get

$$\mathbb{N}_0(T_y < s, \tilde{\ell}_\gamma^y = 0) > 0.$$

Since $\tilde{\ell}_\gamma^y = \ell^y$, this contradicts (7), and this contradiction completes the proof of our claim (8).

Finally, we observe that, for every $s \in (0, 1)$, the law of $(W_r)_{0 \leq r \leq s}$ under $\mathbb{N}_0^{(1)}$ is absolutely continuous with respect to the law of the same process under \mathbb{N}_0 (this is a straightforward consequence of the similar property for the Itô excursion measure and the law of the normalized Brownian excursion; see, e.g., [18], Chapter XII). Hence, (8) also gives, for every $s \in (0, 1)$,

$$\{T_y \leq s\} = \{\tilde{\ell}_s^y > 0\}, \quad \mathbb{N}_0^{(1)} \text{ a.s.},$$

and the fact that the equality in (7) also holds $\mathbb{N}_0^{(1)}$ a.s. readily follows. \square

3. Asymptotics for the range of tree-indexed random walk. Throughout this section, we consider only integers $n \geq 1$ such that $\Pi_\mu(\#\mathcal{T} = n) > 0$ (and when we let $n \rightarrow \infty$, we mean along such values). For every such integer n , let $(\mathcal{T}_n, (Z^n(u))_{u \in \mathcal{T}_n})$ be distributed according to $\Pi_{\mu, \theta}^*(\cdot | \#\mathcal{T} = n)$. Then \mathcal{T}_n is a Galton–Watson tree with offspring distribution μ conditioned to have n vertices, and conditionally on \mathcal{T}_n , $(Z^n(u))_{u \in \mathcal{T}_n}$ is a random walk with jump distribution θ indexed by \mathcal{T}_n .

We set, for every $t > 0$ and $x \in \mathbb{R}^d$,

$$p_t(x) := \frac{1}{(2\pi t)^{d/2} \sqrt{\det M_\theta}} \exp\left(-\frac{x \cdot M_\theta^{-1} x}{2t}\right),$$

where $x \cdot y$ stands for the usual scalar product in \mathbb{R}^d .

For every $a \in \mathbb{Z}^d$, we also set

$$L_n(a) := \sum_{u \in \mathcal{T}_n} \mathbf{1}_{\{Z^n(u) = a\}}.$$

LEMMA 3. *For every $\varepsilon > 0$, there exists a constant C_ε such that, for every n and every $b \in \mathbb{Z}^d$ with $|b| \geq \varepsilon n^{1/4}$,*

$$E[(L_n(b))^2] \leq C_\varepsilon n^{2-d/2}.$$

Furthermore, for every $x, y \in \mathbb{R}^d \setminus \{0\}$, and for every choice of the sequences (x_n) and (y_n) in \mathbb{Z}^d such that $n^{-1/4}x_n \rightarrow x$ and $n^{-1/4}y_n \rightarrow y$ as $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} n^{d/2-2} E[L_n(x_n)L_n(y_n)] = \varphi(x, y),$$

where

$$\begin{aligned} \varphi(x, y) &:= \rho^4 \int_{(\mathbb{R}_+)^3} dr_1 dr_2 dr_3 (r_1 + r_2 + r_3) e^{-\rho^2(r_1+r_2+r_3)^2/2} \\ &\quad \times \int_{\mathbb{R}^d} dz p_{r_1}(z) p_{r_2}(x-z) p_{r_3}(y-z). \end{aligned}$$

The function φ is continuous on $(\mathbb{R}^d \setminus \{0\})^2$.

REMARK. The function φ is in fact continuous on $(\mathbb{R}^d)^2$. Since we will not need this result, we leave the proof to the reader.

PROOF OF LEMMA 3. We first establish the second assertion of the lemma. We let $u_0^n, u_1^n, \dots, u_{n-1}^n$ be the vertices of \mathcal{T}_n listed in lexicographical order. By definition,

$$L_n(x_n) = \sum_{i=0}^{n-1} \mathbf{1}_{\{Z^n(u_i^n)=x_n\}},$$

so that

$$E[L_n(x_n)L_n(y_n)] = E \left[\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \mathbf{1}_{\{Z^n(u_i^n)=x_n, Z^n(u_j^n)=y_n\}} \right].$$

Let H^n be the height function of the tree \mathcal{T}_n , so that $H_i^n = |u_i^n|$ for every $i \in \{0, 1, \dots, n-1\}$. If $i, j \in \{0, 1, \dots, n-1\}$, we also use the notation $\check{H}_{i,j}^n = |u_i^n \wedge u_j^n|$ for the generation of the most recent common ancestor to u_i^n and u_j^n , and note that

$$(9) \quad \left| \check{H}_{i,j}^n - \min_{i \wedge j \leq k \leq i \vee j} H_k^n \right| \leq 1.$$

Write $\pi_k = \theta^{*k}$ for the transition kernels of the random walk with jump distribution θ . By conditioning with respect to the tree \mathcal{T}_n , we get

$$\begin{aligned} (10) \quad & E[L_n(x_n)L_n(y_n)] \\ &= E \left[\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{a \in \mathbb{Z}^d} \pi_{\check{H}_{i,j}^n}^n(a) \pi_{H_i^n - \check{H}_{i,j}^n}^n(x_n - a) \pi_{H_j^n - \check{H}_{i,j}^n}^n(y_n - a) \right] \\ &= n^2 E \left[\int_0^1 \int_0^1 ds dt \Phi_{x_n, y_n}^n(H_{[ns]}^n, H_{[nt]}^n, \check{H}_{[ns], [nt]}^n) \right], \end{aligned}$$

where we have set, for every integers $k, \ell, m \geq 0$ such that $k \wedge \ell \geq m$,

$$\Phi_{x_n, y_n}^n(k, \ell, m) := \sum_{a \in \mathbb{Z}^d} \pi_m(a) \pi_{k-m}(x_n - a) \pi_{\ell-m}(y_n - a).$$

In the remaining part of the proof, we assume that θ is aperiodic [meaning that the subgroup generated by $\{k \geq 0 : \pi_k(0) > 0\}$ is \mathbb{Z}]. Only minor modifications are needed to treat the general case. We can then use the local limit theorem, in a form that can be obtained by combining Theorems 2.3.9 and 2.3.10 in [9]. There exists a sequence δ_n converging to 0 such that, for every $n \geq 1$,

$$(11) \quad \sup_{a \in \mathbb{Z}^d} \left(\left(1 + \frac{|a|^2}{n} \right) n^{d/2} |\pi_n(a) - p_n(a)| \right) \leq \delta_n.$$

Let $(k_n), (\ell_n), (m_n)$ be three sequences of positive integers such that $n^{-1/2}k_n \rightarrow u$, $n^{-1/2}\ell_n \rightarrow v$ and $n^{-1/2}m_n \rightarrow w$, where $0 < w < u \wedge v$. Write

$$\begin{aligned} & n^{d/2} \Phi_{x_n, y_n}^n(k_n, \ell_n, m_n) \\ &= n^{3d/4} \int_{\mathbb{R}^d} dz \pi_{m_n}(\lfloor zn^{1/4} \rfloor) \pi_{k_n - m_n}(x_n - \lfloor zn^{1/4} \rfloor) \pi_{\ell_n - m_n}(y_n - \lfloor zn^{1/4} \rfloor), \end{aligned}$$

and note that, for every fixed $z \in \mathbb{R}^d$,

$$\begin{aligned} & \lim_{n \rightarrow \infty} n^{d/4} \pi_{m_n}(\lfloor zn^{1/4} \rfloor) = p_w(z), \\ & \lim_{n \rightarrow \infty} n^{d/4} \pi_{k_n - m_n}(x_n - \lfloor zn^{1/4} \rfloor) = p_{u-w}(x - z), \\ & \lim_{n \rightarrow \infty} n^{d/4} \pi_{\ell_n - m_n}(y_n - \lfloor zn^{1/4} \rfloor) = p_{v-w}(y - z), \end{aligned}$$

by (11). These convergences even hold uniformly in z . It then follows that

$$(12) \quad \begin{aligned} \lim_{n \rightarrow \infty} n^{d/2} \Phi_{x_n, y_n}^n(k_n, \ell_n, m_n) &= \int_{\mathbb{R}^d} dz p_w(z) p_{u-w}(x - z) p_{v-w}(y - z) \\ &=: \Psi_{x,y}(u, v, w). \end{aligned}$$

Indeed, using (11) again, we have, for every $K > 2(|x| \vee |y|) + 2$ and every sufficiently large n ,

$$\begin{aligned} & n^{3d/4} \int_{\{|z| \geq K+1\}} dz \pi_{m_n}(\lfloor zn^{1/4} \rfloor) \pi_{k_n - m_n}(x_n - \lfloor zn^{1/4} \rfloor) \pi_{\ell_n - m_n}(y_n - \lfloor zn^{1/4} \rfloor) \\ & \leq C \int_{\{|z| \geq K+1\}} dz \left(\frac{1}{(|z| - 1)^2} \right)^3, \end{aligned}$$

with a constant C independent of n and K . The right-hand side of the last display tends to 0 as K tends to infinity. Together with the previously mentioned uniform convergence, this suffices to justify (12).

By [12], Theorem 1.15, we have

$$\left(\frac{\rho}{2} n^{-1/2} H_{\lfloor nt \rfloor}^n \right)_{0 \leq t \leq 1} \xrightarrow[n \rightarrow \infty]{(d)} (\mathbf{e}_t)_{0 \leq t \leq 1},$$

where $(\mathbf{e}_t)_{0 \leq t \leq 1}$ is a normalized Brownian excursion, and we recall that ρ^2 is the variance of μ . The latter convergence holds in the sense of the weak convergence of laws on the Skorokhod space $\mathbb{D}([0, 1], \mathbb{R}_+)$ of càdlàg functions from $[0, 1]$ into \mathbb{R}_+ . Using the Skorokhod representation theorem, we may and will assume that this convergence holds almost surely, uniformly in $t \in [0, 1]$. Recalling (9), it follows that we have also

$$\frac{\rho}{2} n^{-1/2} \check{H}_{[ns], [nt]}^n \xrightarrow{n \rightarrow \infty} \min_{s \wedge t \leq r \leq s \vee t} \mathbf{e}_r =: m_{\mathbf{e}}(s, t),$$

uniformly in $s, t \in [0, 1]$, a.s.

As a consequence of (12) and the preceding observations, we have, for every $s, t \in (0, 1)$ with $s \neq t$,

$$\begin{aligned} (13) \quad & \lim_{n \rightarrow \infty} n^{d/2} \Phi_{x_n, y_n}^n(H_{[ns]}^n, H_{[nt]}^n, \check{H}_{[ns], [nt]}^n) \\ &= \Psi_{x, y} \left(\frac{2}{\rho} \mathbf{e}_s, \frac{2}{\rho} \mathbf{e}_t, \frac{2}{\rho} m_{\mathbf{e}}(s, t) \right), \quad \text{a.s.} \end{aligned}$$

We claim that we can deduce from (10) and (13) that

$$\begin{aligned} (14) \quad & \lim_{n \rightarrow \infty} n^{d/2-2} E[L_n(x_n) L_n(y_n)] \\ &= E \left[\int_0^1 \int_0^1 ds dt \Psi_{x, y} \left(\frac{2}{\rho} \mathbf{e}_s, \frac{2}{\rho} \mathbf{e}_t, \frac{2}{\rho} m_{\mathbf{e}}(s, t) \right) \right]. \end{aligned}$$

Note that the right-hand side of (14) coincides with the function $\varphi(x, y)$ in the lemma. To see this, we can use Theorem III.6 of [11] to verify that the joint density of the triple

$$(m_{\mathbf{e}}(s, t), \mathbf{e}_s - m_{\mathbf{e}}(s, t), \mathbf{e}_t - m_{\mathbf{e}}(s, t))$$

when s and t are chosen uniformly over $[0, 1]$, independently and independently of \mathbf{e} , is

$$16(r_1 + r_2 + r_3) \exp(-2(r_1 + r_2 + r_3)^2).$$

So the proof of the second assertion will be complete if we can justify (14). By Fatou's lemma, (10) and (13), we have first

$$\liminf_{n \rightarrow \infty} n^{d/2-2} E[L_n(x_n) L_n(y_n)] \geq E \left[\int_0^1 \int_0^1 ds dt \Psi_{x, y} \left(\frac{2}{\rho} \mathbf{e}_s, \frac{2}{\rho} \mathbf{e}_t, \frac{2}{\rho} m_{\mathbf{e}}(s, t) \right) \right].$$

Furthermore, dominated convergence shows that, for every $K > 0$,

$$\begin{aligned} & \lim_{n \rightarrow \infty} E \left[\int_0^1 \int_0^1 ds dt (n^{d/2} \Phi_{x_n, y_n}^n(H_{[ns]}^n, H_{[nt]}^n, \check{H}_{[ns], [nt]}^n) \wedge K) \right] \\ &= E \left[\int_0^1 \int_0^1 ds dt \left(\Psi_{x, y} \left(\frac{2}{\rho} \mathbf{e}_s, \frac{2}{\rho} \mathbf{e}_t, \frac{2}{\rho} m_{\mathbf{e}}(s, t) \right) \wedge K \right) \right]. \end{aligned}$$

Write $\Gamma_n(s, t) = n^{d/2} \Phi_{x_n, y_n}^n(H_{[ns]}^n, H_{[nt]}^n, \check{H}_{[ns], [nt]}^n)$ to simplify notation. In view of the preceding comments, it will be enough to verify that

$$(15) \quad \lim_{K \rightarrow \infty} \left(\limsup_{n \rightarrow \infty} E \left[\int_0^1 \int_0^1 ds dt \Gamma_n(s, t) \mathbf{1}_{\{\Gamma_n(s, t) > K\}} \right] \right) = 0.$$

To this end, we will make use of the bound

$$(16) \quad \sup_{k \geq 0} \pi_k(x) \leq M(|x|^{-d} \wedge 1),$$

which holds for every $x \in \mathbb{Z}^d$ with a constant M independent of x . This bound can be obtained easily by combining (11) and Proposition 2.4.6 in [9]. Then let $k, \ell, m \geq 0$ be integers such that $k \wedge \ell \geq m$, and recall that

$$\Phi_{x_n, y_n}^n(k, \ell, m) = \sum_{a \in \mathbb{Z}^d} \pi_m(a) \pi_{k-m}(x_n - a) \pi_{\ell-m}(y_n - a).$$

Fix $\varepsilon > 0$ such that $|x| \wedge |y| > 2\varepsilon$. Consider first the contribution to the sum in the right-hand side coming from values of a such that $|a| \leq \varepsilon n^{1/4}$. For such values of a (and assuming that n is large enough), the estimate (16) allows us to bound both $\pi_{k-m}(x_n - a)$ and $\pi_{\ell-m}(y_n - a)$ by $M\varepsilon^{-d} n^{-d/4}$. On the other hand, if $|a| \geq \varepsilon n^{1/4}$, we can bound $\pi_m(a)$ by $M\varepsilon^{-d} n^{-d/4}$, whereas (11) shows that the sum

$$\sum_{|a| \geq \varepsilon n^{1/4}} \pi_{k-m}(x_n - a) \pi_{\ell-m}(y_n - a)$$

is bounded above by $c_1((k-m)^{-d/2} \wedge (\ell-m)^{-d/2} \wedge 1)$ for some constant c_1 . Summarizing, we get the bound

$$\begin{aligned} \Phi_{x_n, y_n}^n(k, \ell, m) &\leq M^2 \varepsilon^{-2d} n^{-d/2} + c_1 M \varepsilon^{-d} n^{-d/4} ((k-m)^{-d/2} \wedge (\ell-m)^{-d/2} \wedge 1) \\ &\leq c_{1,\varepsilon} n^{-d/2} + c_{2,\varepsilon} n^{-d/4} ((k+\ell-2m)^{-d/2} \wedge 1), \end{aligned}$$

where $c_{1,\varepsilon}$ and $c_{2,\varepsilon}$ are constants that do not depend on n, k, ℓ, m . Then observe that, for every $s, t \in (0, 1)$,

$$H_{[ns]}^n + H_{[nt]}^n - 2\check{H}_{[ns], [nt]}^n = d_n(u_{[ns]}^n, u_{[nt]}^n),$$

where d_n denotes the usual graph distance on \mathcal{T}_n . From the preceding bound, we thus get

$$\Gamma_n(s, t) \leq c_{1,\varepsilon} + c_{2,\varepsilon} n^{d/4} (d_n(u_{[ns]}^n, u_{[nt]}^n)^{-d/2} \wedge 1).$$

It follows that, for every $K > 0$,

$$\begin{aligned}
& \int_0^1 \int_0^1 ds dt \Gamma_n(s, t) \mathbf{1}_{\{\Gamma_n(s, t) > c_{1, \varepsilon} + c_{2, \varepsilon} K\}} \\
& \leq \int_0^1 \int_0^1 ds dt (c_{1, \varepsilon} + c_{2, \varepsilon} n^{d/4} (d_n(u_{[ns]}^n, u_{[nt]}^n)^{-d/2} \wedge 1)) \\
& \quad \times \mathbf{1}_{\{n^{d/4} d_n(u_{[ns]}^n, u_{[nt]}^n)^{-d/2} > K\}} \\
& = n^{-2} \sum_{u, v \in \mathcal{T}_n} (c_{1, \varepsilon} + c_{2, \varepsilon} n^{d/4} (d_n(u, v)^{-d/2} \wedge 1)) \mathbf{1}_{\{d_n(u, v) < K^{-2/d} n^{1/2}\}}.
\end{aligned}$$

By an estimate found in Theorem 1.3 of [5], there exists a constant c_0 that only depends on μ , such that, for every integer $k \geq 1$,

$$(17) \quad E[\#\{(u, v) \in \mathcal{T}_n \times \mathcal{T}_n : d_n(u, v) = k\}] \leq c_0 k n.$$

It then follows that

$$\begin{aligned}
& E \left[\int_0^1 \int_0^1 ds dt \Gamma_n(s, t) \mathbf{1}_{\{\Gamma_n(s, t) > c_{1, \varepsilon} + c_{2, \varepsilon} K\}} \right] \\
& \leq n^{-1} (c_{1, \varepsilon} + c_{2, \varepsilon} n^{d/4}) + c_0 n^{-1} \sum_{k=1}^{\lfloor K^{-2/d} n^{1/2} \rfloor} k (c_{1, \varepsilon} + c_{2, \varepsilon} n^{d/4} k^{-d/2}).
\end{aligned}$$

It is now elementary to verify that the right-hand side of the preceding display has a limit $g(K)$ when $n \rightarrow \infty$, and that $g(K)$ tends to 0 as $K \rightarrow \infty$ (note that we use the fact that $d \leq 3$). This completes the proof of (15) and of the second assertion of the lemma.

The proof of the first assertion is similar and easier. We first note that

$$E[L_n(b)^2] = E \left[\sum_{u, v \in \mathcal{T}_n} \Phi_{b, b}^n(|u|, |v|, |u \wedge v|) \right],$$

where the function $\Phi_{b, b}^n$ is defined as above. Then, assuming that $|b| \geq 2\varepsilon n^{1/4}$, the same arguments as in the first part of the proof give the bound

$$\Phi_{b, b}^n(|u|, |v|, |u \wedge v|) \leq c_{1, \varepsilon} n^{-d/2} + c_{2, \varepsilon} n^{-d/4} (d_n(u, v)^{-d/2} \wedge 1).$$

By summing over all choices of u and v , it follows that

$$\begin{aligned}
& E[L_n(b)^2] \\
& \leq c_{1, \varepsilon} n^{2-d/2} \\
& \quad + c_{2, \varepsilon} n^{-d/4} \left(n + E \left[\sum_{u, v \in \mathcal{T}_n, 1 \leq d_n(u, v) \leq \sqrt{n}} d_n(u, v)^{-d/2} \right] + n^2 \times n^{-d/4} \right)
\end{aligned}$$

$$\begin{aligned} &\leq (c_{1,\varepsilon} + 2c_{2,\varepsilon})n^{2-d/2} \\ &\quad + c_{2,\varepsilon}n^{-d/4} \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} k^{-d/2} E[\#\{(u,v) \in \mathcal{T}_n : d_n(u,v) = k\}], \end{aligned}$$

and the bound stated in the first assertion easily follows from (17).

Let us finally establish the continuity of φ . We fix $\varepsilon > 0$ and verify that φ is continuous on the set $\{|x| \geq 2\varepsilon, |y| \geq 2\varepsilon\}$. We split the integral in dz in two parts:

– The integral over $|z| \leq \varepsilon$. Write $\varphi_{1,\varepsilon}(x,y)$ for the contribution of this integral. We observe that, if $|z| \leq \varepsilon$, the function $x \mapsto p_{r_2}(x-z)$ is Lipschitz uniformly in z and in r_2 on the set $\{|x| \geq 2\varepsilon\}$, and a similar property holds for the function $y \mapsto p_{r_3}(y-z)$. It follows that $\varphi_{1,\varepsilon}$ is a Lipschitz function of (x,y) on the set $\{|x| \geq 2\varepsilon, |y| \geq 2\varepsilon\}$.

– The integral over $|z| > \varepsilon$. Write $\varphi_{2,\varepsilon}(x,y)$ for the contribution of this integral. Note that if $(u_n, v_n)_{n \geq 1}$ is a sequence in $\mathbb{R}^d \times \mathbb{R}^d$ such that $|u_n| \wedge |v_n| \geq 2\varepsilon$ for every n , and (u_n, v_n) converges to (x, y) as $n \rightarrow \infty$, we have, for every fixed $r_1, r_2, r_3 > 0$,

$$\begin{aligned} &\int_{\{|z| > \varepsilon\}} dz p_{r_1}(z) p_{r_2}(u_n - z) p_{r_3}(v_n - z) \\ &\quad \xrightarrow{n \rightarrow \infty} \int_{\{|z| > \varepsilon\}} dz p_{r_1}(z) p_{r_2}(x - z) p_{r_3}(y - z). \end{aligned}$$

We can then use dominated convergence, since there exist constants c_ε and \tilde{c}_ε that depend only on ε , such that

$$\int_{\{|z| > \varepsilon\}} dz p_{r_1}(z) p_{r_2}(u_n - z) p_{r_3}(v_n - z) \leq c_\varepsilon p_{r_2+r_3}(u_n - v_n) \leq \tilde{c}_\varepsilon (r_2 + r_3)^{-d/2},$$

and the right-hand side is integrable for the measure $(r_1 + r_2 + r_3) \times e^{-\rho^2(r_1+r_2+r_3)^2/2} dr_1 dr_2 dr_3$. It follows that $\varphi_{2,\varepsilon}$ is also continuous on the set $\{|x| \geq 2\varepsilon, |y| \geq 2\varepsilon\}$.

The preceding considerations complete the proof. \square

In what follows, we use the notation $W^{(1)} = (W_s^{(1)})_{0 \leq s \leq 1}$ for a process distributed according to $\mathbb{N}_0^{(1)}$. We recall a result of Janson and Marckert [6] that will play an important role below. As in the proof of Lemma 3, we let $u_0^n, u_1^n, \dots, u_{n-1}^n$ be the vertices of \mathcal{T}_n listed in lexicographical order. For every $j \in \{0, 1, \dots, n-1\}$ write $Z_j^n = Z^n(u_j^n)$ for the spatial location of u_j^n , and $Z_n^n = 0$ by convention. Recalling our assumption (2), we get from [6], Theorem 2, that

$$(18) \quad \left(\sqrt{\frac{\rho}{2}} n^{-1/4} Z_{[nt]}^n \right)_{0 \leq t \leq 1} \xrightarrow[n \rightarrow \infty]{(d)} (M_\theta^{1/2} \widehat{W}_t^{(1)})_{0 \leq t \leq 1},$$

where as usual $M_\theta^{1/2}$ is the unique positive definite symmetric matrix such that $M_\theta = (M_\theta^{1/2})^2$, and the convergence holds in distribution in the Skorokhod space $\mathbb{D}([0, 1], \mathbb{R}^d)$. Note that there are two minor differences between [6] and the present setting. First, [6] considers one-dimensional labels, whereas our spatial locations take values in \mathbb{Z}^d . However, we can simply project $Z_n(u)$ on the coordinate axes to get tightness in the convergence (18) from the results of [6], and convergence of finite-dimensional marginals is easy just as in [6], Proof of Theorem 1. Second, the “discrete snake” of [6] lists the labels encountered when exploring the tree \mathcal{T}_n in depth first traversal (or contour order), whereas we are here enumerating the vertices in lexicographical order. Nevertheless, the very same arguments that are used to relate the contour process and the height function of a random tree (see [16] or [12], Section 1.6) show that asymptotics for the discrete snakes of [6] imply similar asymptotics for the labels listed in lexicographical order of vertices.

In the next theorem, the notation $(l^x, x \in \mathbb{R}^d)$ stands for the collection of local times of $W^{(1)}$, which are defined as the continuous density of the occupation measure of $W^{(1)}$ as in Proposition 1. We define a constant $c > 0$ by setting

$$(19) \quad c := \frac{1}{\sigma} \sqrt{\frac{\rho}{2}},$$

where $\sigma^2 = (\det M_\theta)^{1/d}$ as previously. We also use the notation $M_\theta^{-1/2} = (M_\theta^{1/2})^{-1}$.

THEOREM 4. *Let $x^1, \dots, x^p \in \mathbb{R}^d \setminus \{0\}$, and let $(x_n^1), \dots, (x_n^p)$ be sequences in \mathbb{Z}^d such that $\sqrt{\frac{\rho}{2}} n^{-1/4} M_\theta^{-1/2} x_n^j \rightarrow x^j$ as $n \rightarrow \infty$, for every $1 \leq j \leq p$. Then*

$$(n^{d/4-1} L_n(x_n^1), \dots, n^{d/4-1} L_n(x_n^p)) \xrightarrow[n \rightarrow \infty]{(d)} (c^d l^{x^1}, \dots, c^d l^{x^p}),$$

where the constant c is given by (19).

REMARKS. (i) As mentioned in the [Introduction](#), this result should be compared with Theorem 1 in [7], which deals with local times of branching random walk in \mathbb{Z}^d for $d = 2$ or 3 . See also [3], Theorem 3.6 and [5], Theorem 1.1, for stronger versions of the convergence in Theorem 4 when $d = 1$.

(ii) It is likely that the result of Lemma 3 still holds when $x = 0$ or $y = 0$, and then the condition $x^i \neq 0$ in the preceding theorem could be removed, using also the remark after Lemma 3. Proving this reinforcement of Lemma 3

would however require additional technicalities. Since this extension is not needed in the proof of our main results, we will not address this problem here.

PROOF OF THEOREM 4. To simplify the presentation, we give the details of the proof only in the isotropic case where $M_\theta = \sigma^2 \text{Id}$ (the non-isotropic case is treated in exactly the same manner at the cost of a somewhat heavier notation). Our condition on the sequences (x_n^j) then just says that $cn^{-1/4}x_n^j \rightarrow x^j$ as $n \rightarrow \infty$.

By the Skorokhod representation theorem, we may and will assume that the convergence (18) holds a.s. To obtain the result of the theorem, it is then enough to verify that, if $x \in \mathbb{R}^d \setminus \{0\}$ and (x_n) is a sequence in \mathbb{Z}^d such that $cn^{-1/4}x_n \rightarrow x$ as $n \rightarrow \infty$, we have

$$(20) \quad n^{d/4-1}L_n(x_n) \xrightarrow[n \rightarrow \infty]{(P)} c^d l^x.$$

To this end, fix x and the sequence (x_n) , and for every $\varepsilon \in (0, |x|)$, let g_ε be a nonnegative continuous function on \mathbb{R}^d , with compact support contained in the open ball of radius ε centered at x , and such that

$$\int_{\mathbb{R}^d} g_\varepsilon(y) dy = 1.$$

It follows from (18) (which we assume to hold a.s.) that, for every fixed $\varepsilon \in (0, |x|)$,

$$\int_0^1 g_\varepsilon(cn^{-1/4}Z_{[nt]}^n) dt \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \int_0^1 g_\varepsilon(\widehat{W}_t^{(1)}) dt.$$

Furthermore,

$$\int_0^1 g_\varepsilon(\widehat{W}_t^{(1)}) dt = \int_{\mathbb{R}^d} g_\varepsilon(y) l^y dy \xrightarrow[\varepsilon \rightarrow 0]{\text{a.s.}} l^x,$$

by the continuity of local times. Let $\delta > 0$. By combining the last two convergences, we can find $\varepsilon_1 \in (0, |x|)$ such that, for every $\varepsilon \in (0, \varepsilon_1)$, there exists an integer $n_1(\varepsilon)$ so that for every $n \geq n_1(\varepsilon)$,

$$(21) \quad P\left(\left|\int_0^1 g_\varepsilon(cn^{-1/4}Z_{[nt]}^n) dt - l^x\right| > \delta\right) < \delta.$$

However, we have

$$\begin{aligned} \int_0^1 g_\varepsilon(cn^{-1/4}Z_{[nt]}^n) dt &= \frac{1}{n} \sum_{a \in \mathbb{Z}^d} g_\varepsilon(cn^{-1/4}a) L_n(a) \\ &= n^{d/4-1} \int_{\mathbb{R}^d} g_\varepsilon(cn^{-1/4}\lfloor n^{1/4}y \rfloor) L_n(\lfloor n^{1/4}y \rfloor) dy. \end{aligned}$$

Set

$$\eta_n(\varepsilon) := \int_{\mathbb{R}^d} g_\varepsilon(cn^{-1/4}\lfloor n^{1/4}y \rfloor) dy$$

and note that

$$\eta_n(\varepsilon) \xrightarrow{n \rightarrow \infty} \int_{\mathbb{R}^d} g_\varepsilon(cy) dy = c^{-d}.$$

By the Cauchy–Schwarz inequality,

$$\begin{aligned} & E \left[\left(\int_0^1 g_\varepsilon(cn^{-1/4}Z_{\lfloor nt \rfloor}^n) dt - \eta_n(\varepsilon)n^{d/4-1}L_n(x_n) \right)^2 \right] \\ &= E \left[\left(n^{d/4-1} \int_{\mathbb{R}^d} g_\varepsilon(cn^{-1/4}\lfloor n^{1/4}y \rfloor) (L_n(\lfloor n^{1/4}y \rfloor) - L_n(x_n)) dy \right)^2 \right] \\ &\leq \eta_n(\varepsilon) \times n^{d/2-2} \int_{\mathbb{R}^d} dy g_\varepsilon(cn^{-1/4}\lfloor n^{1/4}y \rfloor) E[(L_n(\lfloor n^{1/4}y \rfloor) - L_n(x_n))^2]. \end{aligned}$$

Using the first assertion of Lemma 3, one easily gets that, for every fixed $\varepsilon \in (0, |x|)$,

$$n^{d/2-2} \int_{\mathbb{R}^d} dy |g_\varepsilon(cn^{-1/4}\lfloor n^{1/4}y \rfloor) - g_\varepsilon(cy)| E[(L_n(\lfloor n^{1/4}y \rfloor) - L_n(x_n))^2] \xrightarrow{n \rightarrow \infty} 0.$$

On the other hand, by the second assertion of the lemma,

$$\begin{aligned} & n^{d/2-2} \int_{\mathbb{R}^d} dy g_\varepsilon(cy) E[(L_n(\lfloor n^{1/4}y \rfloor) - L_n(x_n))^2] \\ & \xrightarrow{n \rightarrow \infty} \int_{\mathbb{R}^d} dy g_\varepsilon(cy) \left(\varphi(y, y) - 2\varphi\left(\frac{x}{c}, y\right) + \varphi\left(\frac{x}{c}, \frac{x}{c}\right) \right). \end{aligned}$$

If γ_ε stands for the limit in the last display, the continuity of φ ensures that γ_ε tends to 0 as $\varepsilon \rightarrow 0$.

From the preceding considerations, we have

$$\limsup_{n \rightarrow \infty} E \left[\left(\int_0^1 g_\varepsilon(cn^{-1/4}Z_{\lfloor nt \rfloor}^n) dt - \eta_n(\varepsilon)n^{d/4-1}L_n(x_n) \right)^2 \right] \leq c^{-d}\gamma_\varepsilon.$$

Hence, we can find $\varepsilon_2 \in (0, |x|)$ small enough so that, for every $\varepsilon \in (0, \varepsilon_2)$, there exists an integer $n_2(\varepsilon)$ such that, for every $n \geq n_2(\varepsilon)$,

$$(22) \quad P \left(\left| \int_0^1 g_\varepsilon(cn^{-1/4}Z_{\lfloor nt \rfloor}^n) dt - \eta_n(\varepsilon)n^{d/4-1}L_n(x_n) \right| > \delta \right) < \delta.$$

By combining (21) and (22), we see that, for every $\varepsilon \in (0, \varepsilon_1 \wedge \varepsilon_2)$ and $n \geq n_1(\varepsilon) \vee n_2(\varepsilon)$,

$$P(|\eta_n(\varepsilon)n^{d/4-1}L_n(x_n) - l^x| > 2\delta) < 2\delta.$$

Our claim (20) easily follows, since $\eta_n(\varepsilon)$ tends to c^{-d} as $n \rightarrow \infty$. \square

Set $R_n = \#\{Z^n(u) : u \in \mathcal{T}_n\}$. Recall the constant c from (19), and also recall that λ_d denotes Lebesgue measure on \mathbb{R}^d .

THEOREM 5. *We have*

$$n^{-d/4} R_n \xrightarrow[n \rightarrow \infty]{(d)} c^{-d} \lambda_d(\mathcal{S}),$$

where \mathcal{S} stands for the support of ISE.

PROOF. Again, for the sake of simplicity, we give details only in the isotropic case $M_\theta = \sigma^2 \text{Id}$. From the definition of ISE, we may take $\mathcal{S} = \{\widehat{W}_t^{(1)} : 0 \leq t \leq 1\}$. We then set, for every $\varepsilon > 0$,

$$\mathcal{S}_\varepsilon := \{x \in \mathbb{R}^d : \text{dist}(x, \mathcal{S}) \leq \varepsilon\}.$$

As in the preceding proof, we may and will assume that the convergence (18) holds almost surely. It then follows that, for every $\varepsilon > 0$,

$$P(\{cn^{-1/4}Z^n(u) : u \in \mathcal{T}_n\} \subset \mathcal{S}_\varepsilon) \xrightarrow[n \rightarrow \infty]{} 1.$$

Fix $K > 0$, and let $B(0, K)$ stand for the closed ball of radius K centered at 0 in \mathbb{R}^d . Also set $\mathcal{S}_\varepsilon^{(K)} := \mathcal{S}_\varepsilon \cap B(0, K + \varepsilon)$. It follows that we have also

$$P(\{Z^n(u) : u \in \mathcal{T}_n\} \cap B(0, c^{-1}n^{1/4}K)) \subset c^{-1}n^{1/4}\mathcal{S}_\varepsilon^{(K)} \xrightarrow[n \rightarrow \infty]{} 1.$$

Applying the latter convergence with ε replaced by $\varepsilon/2$, we get

$$P(\#(\{Z^n(u) : u \in \mathcal{T}_n\} \cap B(0, c^{-1}n^{1/4}K)) \leq c^{-d}n^{d/4}\lambda_d(\mathcal{S}_\varepsilon^{(K)})) \xrightarrow[n \rightarrow \infty]{} 1.$$

Write $R_n^{(K)} := \#(\{Z^n(u) : u \in \mathcal{T}_n\} \cap B(0, c^{-1}n^{1/4}K))$. Since $\lambda_d(\mathcal{S}_\varepsilon^{(K)}) \downarrow \lambda_d(\mathcal{S} \cap B(0, K))$ as $\varepsilon \downarrow 0$, we obtain that, for every $\delta > 0$,

$$P(n^{-d/4}R_n^{(K)} \leq c^{-d}\lambda_d(\mathcal{S} \cap B(0, K)) + \delta) \xrightarrow[n \rightarrow \infty]{} 1,$$

and, therefore, since the variables $n^{-d/4}R_n^{(K)}$ are uniformly bounded,

$$(23) \quad \lim_{n \rightarrow \infty} E[(n^{-d/4}R_n^{(K)} - c^{-d}\lambda_d(\mathcal{S} \cap B(0, K)))^+] = 0.$$

On the other hand, we claim that we have also

$$(24) \quad \liminf_{n \rightarrow \infty} E[n^{-d/4}R_n^{(K)}] \geq c^{-d}E[\lambda_d(\mathcal{S} \cap B(0, K))].$$

To see this, observe that

$$\begin{aligned}
E[R_n^{(K)}] &= \sum_{a \in \mathbb{Z}^d \cap B(0, c^{-1}n^{1/4}K)} P(L_n(a) > 0) \\
&= \int_{B(0, c^{-1}n^{1/4}K)} dx P(L_n(\lfloor x \rfloor) > 0) + O(n^{(d-1)/4}) \\
&= n^{d/4} \int_{B(0, c^{-1}K)} dy P(L_n(\lfloor n^{1/4}y \rfloor) > 0) + O(n^{(d-1)/4})
\end{aligned}$$

as $n \rightarrow \infty$. By Theorem 4, for every $y \neq 0$,

$$\liminf_{n \rightarrow \infty} P(L_n(\lfloor n^{1/4}y \rfloor) > 0) \geq P(l^{cy} > 0) = P(cy \in \mathcal{S}),$$

where the equality is derived from Proposition 2. Fatou's lemma then gives

$$\liminf_{n \rightarrow \infty} n^{-d/4} E[R_n^{(K)}] \geq \int_{B(0, c^{-1}K)} dy P(cy \in \mathcal{S}) = c^{-d} E[\lambda_d(\mathcal{S} \cap B(0, K))],$$

which completes the proof of (24).

Using the trivial identity $|x| = 2x^+ - x$ for every real x , we deduce from (23) and (24) that

$$\lim_{n \rightarrow \infty} E[|n^{-d/4} R_n^{(K)} - c^{-d} \lambda_d(\mathcal{S} \cap B(0, K))|] = 0.$$

However, we see from (18) that, for every $\delta > 0$, we can choose K sufficiently large so that we have both $P(\mathcal{S} \subset B(0, K)) \geq 1 - \delta$ and $P(R_n^{(K)} = R_n) \geq 1 - \delta$ for every integer n . It then follows from the previous convergence that $n^{-d/4} R_n$ converges in probability to $c^{-d} \lambda_d(\mathcal{S})$ as $n \rightarrow \infty$, and this completes the proof of Theorem 5. \square

4. Branching random walk. We will now discuss similar results for branching random walk in \mathbb{Z}^d . We consider a system of particles in \mathbb{Z}^d that evolves in discrete time in the following way. At time $n = 0$, there are p particles all located at the origin of \mathbb{Z}^d (we will comment on more general initial configurations in Section 5.3). A particle located at the site $a \in \mathbb{Z}^d$ at time n gives rise at time $n + 1$ to a random number of offspring distributed according to μ , and their locations are obtained by adding to a (independently for each offspring) a spatial displacement distributed according to θ .

In a more formal way, we consider p independent random spatial trees

$$(\mathcal{T}^{(1)}, (Z^{(1)}(u))_{u \in \mathcal{T}^{(1)}}), \dots, (\mathcal{T}^{(p)}, (Z^{(p)}(u))_{u \in \mathcal{T}^{(p)}})$$

distributed according to $\Pi_{\mu, \theta}^*$, and, for every integer $n \geq 0$, we consider the random point measure

$$X_n^{[p]} := \sum_{j=1}^p \left(\sum_{u \in \mathcal{T}^{(j)}, |u|=n} \delta_{Z^{(j)}(u)} \right),$$

which corresponds to the sum of the Dirac point masses at the positions of all particles alive at time n .

The set $\mathcal{V}^{[p]}$ of all sites visited by the particles is the union over all $n \geq 0$ of the supports of $X_n^{[p]}$, or equivalently

$$\mathcal{V}^{[p]} = \{a \in \mathbb{Z}^d : a = Z^{(j)}(u) \text{ for some } j \in \{1, \dots, p\} \text{ and } u \in \mathcal{T}^{(j)}\}.$$

In a way similar to Theorem 5, we are interested in limit theorems for $\#\mathcal{V}^{[p]}$ when $p \rightarrow \infty$. To this end, we will first state an analog of the convergence (18). For every $j \in \{1, \dots, p\}$, let

$$\emptyset = u_0^{(j)} \prec u_1^{(j)} \prec \dots \prec u_{\#\mathcal{T}^{(j)}-1}^{(j)}$$

be the vertices of $\mathcal{T}^{(j)}$ listed in lexicographical order, and set $H_i^{(j)} = |u_i^{(j)}|$ and $Z_i^{(j)} = Z^{(j)}(u_i^{(j)})$, for $0 \leq i \leq \#\mathcal{T}^{(j)} - 1$. Define the height function $(H_k^{[p]}, k \geq 0)$ of $\mathcal{T}^{(1)}, \dots, \mathcal{T}^{(p)}$ by concatenating the discrete functions $(H_i^{(j)}, 0 \leq i \leq \#\mathcal{T}^{(j)} - 1)$, and setting $H_k^{[p]} = 0$ for $k \geq \#\mathcal{T}^{(1)} + \dots + \#\mathcal{T}^{(p)}$. Similarly, define the function $(Z_k^{[p]}, k \geq 0)$ by concatenating the discrete functions $(Z_i^{(j)}, 0 \leq i \leq \#\mathcal{T}^{(j)} - 1)$, and setting $Z_k^{[p]} = 0$ for $k \geq \#\mathcal{T}^{(1)} + \dots + \#\mathcal{T}^{(p)}$. Finally, we use linear interpolation to define $H_t^{[p]}$ and $Z_t^{[p]}$ for every real $t \geq 0$. We can now state our analog of (18).

PROPOSITION 6. *We have*

$$\left(\left(\frac{\rho}{2} p^{-1} H_{p^2 s}^{[p]}, \sqrt{\frac{\rho}{2}} p^{-1/2} Z_{p^2 s}^{[p]} \right)_{s \geq 0}, p^{-2} (\#\mathcal{T}^{(1)} + \dots + \#\mathcal{T}^{(p)}) \right) \\ \xrightarrow[p \rightarrow \infty]{(d)} ((\zeta_{s \wedge \tau}, M_\theta^{1/2} \widehat{W}_{s \wedge \tau})_{s \geq 0}, \tau),$$

where $(W_s)_{s \geq 0}$ is a standard Brownian snake, τ denotes the first hitting time of $2/\rho$ by the local time at 0 of the lifetime process $(\zeta_s)_{s \geq 0}$, and the convergence of processes holds in the sense of the topology of uniform convergence on compact sets.

The joint convergence of the processes $\frac{\rho}{2} p^{-1} H_{p^2 s}^{[p]}$ and of the random variables $p^{-2} (\#\mathcal{T}^{(1)} + \dots + \#\mathcal{T}^{(p)})$ is a consequence of [12], Theorem 1.8, see in particular (7) and (9) in [12] (note that the local times of the process $(\zeta_s)_{s \geq 0}$ are chosen to be right-continuous in the space variable, so that our local time at 0 is twice the local time that appears in the display (7) in [12]). Given the latter joint convergence, the desired statement can be obtained by following the arguments of the proof of Theorem 2 in [6]. The fact that we are dealing with unconditioned trees makes things easier than in [6] and we omit the details.

We now state an intermediate result, which is of independent interest. Under the probability measure $\Pi_{\mu,\theta}^*$, we let $\mathbf{R} := \{z_u : u \in \mathcal{T}\}$ be the set of all points visited by the tree-indexed random walk.

THEOREM 7. *We have*

$$\lim_{|a| \rightarrow \infty} |M_\theta^{-1/2} a|^2 \Pi_{\mu,\theta}^*(a \in \mathbf{R}) = \frac{2(4-d)}{\rho^2}.$$

PROOF. We start by proving the upper bound

$$\limsup_{|a| \rightarrow \infty} |M_\theta^{-1/2} a|^2 \Pi_{\mu,\theta}^*(a \in \mathbf{R}) \leq \frac{2(4-d)}{\rho^2}.$$

By an easy compactness argument, it is enough to prove that, if (a_k) is a sequence in \mathbb{Z}^d such that $|a_k| \rightarrow \infty$ and $a_k/|a_k| \rightarrow x$, with $x \in \mathbb{R}^d$ and $|x| = 1$, then

$$(25) \quad \limsup_{k \rightarrow \infty} |a_k|^2 \Pi_{\mu,\theta}^*(a_k \in \mathbf{R}) \leq \frac{2(4-d)}{\rho^2 |M_\theta^{-1/2} x|^2}.$$

Set $p_k = |a_k|^2 \in \mathbb{Z}_+$ to simplify notation. We note that

$$(26) \quad P(a_k \in \mathcal{V}^{[p_k]}) = 1 - (1 - \Pi_{\mu,\theta}^*(a_k \in \mathbf{R}))^{p_k}.$$

On the other hand, it follows from our definitions that

$$P(a_k \in \mathcal{V}^{[p_k]}) \leq P\left(\exists s \geq 0 : \frac{1}{\sqrt{p_k}} Z_s^{(p_k)} = \frac{a_k}{|a_k|}\right).$$

We can then use Proposition 6 to get

$$\begin{aligned} \limsup_{k \rightarrow \infty} P(a_k \in \mathcal{V}^{[p_k]}) &\leq \mathbb{P}_0\left(\exists s \in [0, \tau] : M_\theta^{1/2} \widehat{W}_s = \sqrt{\frac{\rho}{2}} x\right) \\ &= 1 - \exp\left(-\frac{2}{\rho} \mathbb{N}_0\left(\sqrt{\frac{\rho}{2}} M_\theta^{-1/2} x \in \mathcal{R}\right)\right) \\ &= 1 - \exp\left(-\frac{2(4-d)}{\rho^2 |M_\theta^{-1/2} x|^2}\right). \end{aligned}$$

The second line follows from excursion theory for the Brownian snake, and the third one uses the formula for $\mathbb{N}_0(y \in \mathcal{R})$, which has been recalled already in the proof of Proposition 2. By combining the bound of the last display with (26), we get our claim (25), and this completes the proof of the upper bound.

Let us turn to the proof of the lower bound. As in the proof of the upper bound, it is enough to consider a sequence (a_k) in \mathbb{Z}^d such that $|a_k| \rightarrow \infty$ and $a_k/|a_k| \rightarrow x$, with $x \in \mathbb{R}^d$ and $|x| = 1$, and then to verify that

$$(27) \quad \liminf_{k \rightarrow \infty} |a_k|^2 \Pi_{\mu, \theta}^*(a_k \in \mathbf{R}) \geq \frac{2(4-d)}{\rho^2 |M_\theta^{-1/2} x|^2}.$$

As previously, we set $p_k = |a_k|^2$. We fix $0 < \varepsilon < M$, and we introduce the function g_μ defined on \mathbb{Z}_+ by $g_\mu(j) = \Pi_\mu(\#\mathcal{T} = j)$. Then

$$\begin{aligned} |a_k|^2 \Pi_{\mu, \theta}^*(a_k \in \mathbf{R}) &\geq p_k^3 \int_\varepsilon^M dr \Pi_{\mu, \theta}^*(a_k \in \mathbf{R}, \#\mathcal{T} = \lfloor p_k^2 r \rfloor) \\ &= p_k^3 \int_\varepsilon^M dr g_\mu(\lfloor p_k^2 r \rfloor) P(L_{\lfloor p_k^2 r \rfloor}(a_k) > 0), \end{aligned}$$

where we use the same notation as in Lemma 3: $L_n(b)$ denotes the number of visits of site b by a random walk indexed by a tree distributed according to $\Pi_\mu(\cdot | \#\mathcal{T} = n)$. Note that Theorem 4 gives, for every $r \in [\varepsilon, M]$,

$$\liminf_{k \rightarrow \infty} P(L_{\lfloor p_k^2 r \rfloor}(a_k) > 0) \geq P(l^{r^{-1/4}z} > 0),$$

where we write $z = \sqrt{\frac{\rho}{2}} M_\theta^{-1/2} x$ to simplify notation. To complete the argument, we consider for simplicity the aperiodic case where μ is not supported on a strict subgroup of \mathbb{Z} [the reader will easily be able to extend our method to the general case, using (3) instead of (4)]. By (4), we have for every $r \in [\varepsilon, M]$,

$$\lim_{k \rightarrow \infty} p_k^3 g_\mu(\lfloor p_k^2 r \rfloor) = \frac{1}{\rho \sqrt{2\pi r^3}}.$$

Using this together with the preceding display, and applying Fatou's lemma, we obtain

$$(28) \quad \liminf_{k \rightarrow \infty} |a_k|^2 \Pi_{\mu, \theta}^*(a_k \in \mathbf{R}) \geq \int_\varepsilon^M \frac{dr}{\rho \sqrt{2\pi r^3}} P(l^{r^{-1/4}z} > 0).$$

A scaling argument shows that

$$P(l^{r^{-1/4}z} > 0) = \mathbb{N}_0^{(1)}(\ell^{r^{-1/4}z} > 0) = \mathbb{N}_0^{(r)}(\ell^z > 0).$$

Using this remark and formula (5), we see that the right-hand side of (28) can be rewritten as $\frac{2}{\rho} \mathbb{N}_0(\mathbf{1}_{\{\varepsilon < \gamma < M\}} \mathbf{1}_{\{\ell^z > 0\}})$. By choosing ε small enough and M large enough, the latter quantity can be made arbitrarily close to

$$\frac{2}{\rho} \mathbb{N}_0(\ell^z > 0) = \frac{2}{\rho} \left(2 - \frac{d}{2} \right) |z|^{-2} = \frac{2(4-d)}{\rho^2 |M_\theta^{-1/2} x|^2}.$$

This completes the proof of the lower bound and of Theorem 7. \square

Recall our notation $\mathcal{V}^{[p]}$ for the set of all sites visited by the branching random walk starting with p initial particles located at the origin.

THEOREM 8. *We have*

$$p^{-d/2} \# \mathcal{V}^{[p]} \xrightarrow[p \rightarrow \infty]{(d)} \left(\frac{2\sigma}{\rho} \right)^d \lambda_d \left(\bigcup_{t \geq 0} \text{supp } X_t \right),$$

where $(X_t)_{t \geq 0}$ is a d -dimensional super-Brownian motion with branching mechanism $\psi(u) = 2u^2$ started from δ_0 , and $\text{supp } X_t$ denotes the topological support of X_t .

PROOF. Via the Skorokhod representation theorem, we may and will assume that the convergence in Proposition 6 holds a.s., and we will then prove that the convergence of the theorem holds in probability. If $\varepsilon > 0$ is fixed, the (a.s.) convergence in Proposition 6 implies that, a.s. for all large enough p , we have

$$\sqrt{\frac{\rho}{2}} p^{-1/2} \mathcal{V}^{[p]} \subset \mathcal{U}_\varepsilon(\{M_\theta^{1/2} \widehat{W}_s : 0 \leq s \leq \tau\}),$$

where, for any compact subset \mathcal{K} of \mathbb{R}^d , $\mathcal{U}_\varepsilon(\mathcal{K})$ denotes the set of all points whose distance from \mathcal{K} is strictly less than ε . It follows that we have a.s.

$$\limsup_{p \rightarrow \infty} p^{-d/2} \# \mathcal{V}^{[p]} \leq \left(\frac{2}{\rho} \right)^{d/2} \lambda_d(\mathcal{U}_{2\varepsilon}(\{M_\theta^{1/2} \widehat{W}_s : 0 \leq s \leq \tau\})).$$

Since ε was arbitrary, we also get a.s.

$$(29) \quad \limsup_{p \rightarrow \infty} p^{-d/2} \# \mathcal{V}^{[p]} \leq \left(\frac{2}{\rho} \right)^{d/2} \lambda_d(\{M_\theta^{1/2} \widehat{W}_s : 0 \leq s \leq \tau\}).$$

To get an estimate in the reverse direction, we argue in a way very similar to the proof of Theorem 5. We fix $K > 0$, and note that a minor modification of the preceding arguments also gives a.s.

$$\begin{aligned} \limsup_{p \rightarrow \infty} p^{-d/2} \# (\mathcal{V}^{[p]} \cap B(0, p^{1/2} K)) \\ \leq \left(\frac{2}{\rho} \right)^{d/2} \lambda_d(\{M_\theta^{1/2} \widehat{W}_s : 0 \leq s \leq \tau\} \cap B(0, K')), \end{aligned}$$

where $K' = \sqrt{\frac{\rho}{2}}K$. Since the variables $p^{-d/2}\#(\mathcal{V}^{[p]} \cap B(0, p^{1/2}K))$ are uniformly bounded, it follows that

$$(30) \quad \lim_{p \rightarrow \infty} E \left[\left(p^{-d/2}\#(\mathcal{V}^{[p]} \cap B(0, p^{1/2}K)) - \left(\frac{2}{\rho} \right)^{d/2} \lambda_d(\{M_\theta^{1/2}\widehat{W}_s : 0 \leq s \leq \tau\} \cap B(0, K')) \right)^+ \right] = 0.$$

On the other hand,

$$\begin{aligned} & p^{-d/2} E[\#(\mathcal{V}^{[p]} \cap B(0, p^{1/2}K))] \\ &= p^{-d/2} \sum_{a \in \mathbb{Z}^d \cap B(0, p^{1/2}K)} P(a \in \mathcal{V}^{[p]}) \\ &= p^{-d/2} \sum_{a \in \mathbb{Z}^d \cap B(0, p^{1/2}K)} (1 - (1 - \Pi_{\mu, \theta}^*(a \in \mathbf{R}))^p) \\ &\xrightarrow{p \rightarrow \infty} \int_{B(0, K)} dx \left(1 - \exp\left(-\frac{2(4-d)}{\rho^2 |M_\theta^{-1/2}x|^2} \right) \right), \end{aligned}$$

where the last line is an easy consequence of Theorem 7. Furthermore,

$$\begin{aligned} & E \left[\left(\frac{2}{\rho} \right)^{d/2} \lambda_d(\{M_\theta^{1/2}\widehat{W}_s : 0 \leq s \leq \tau\} \cap B(0, K')) \right] \\ &= \left(\frac{2}{\rho} \right)^{d/2} \int_{B(0, K')} dy \left(1 - \exp\left(-\frac{2}{\rho} \mathbb{N}_0(M_\theta^{-1/2}y \in \mathcal{R}) \right) \right) \\ &= \left(\frac{2}{\rho} \right)^{d/2} \int_{B(0, K')} dy \left(1 - \exp\left(-\frac{4-d}{\rho |M_\theta^{-1/2}y|^2} \right) \right) \\ &= \int_{B(0, K)} dx \left(1 - \exp\left(-\frac{2(4-d)}{\rho^2 |M_\theta^{-1/2}x|^2} \right) \right). \end{aligned}$$

From the last two displays, we get

$$(31) \quad \begin{aligned} & \lim_{p \rightarrow \infty} E[p^{-d/2}\#(\mathcal{V}^{[p]} \cap B(0, p^{1/2}K))] \\ &= E \left[\left(\frac{2}{\rho} \right)^{d/2} \lambda_d(\{M_\theta^{1/2}\widehat{W}_s : 0 \leq s \leq \tau\} \cap B(0, K')) \right]. \end{aligned}$$

From (30) and (31), we have

$$\lim_{p \rightarrow \infty} E \left[\left| p^{-d/2}\#(\mathcal{V}^{[p]} \cap B(0, p^{1/2}K)) \right. \right]$$

$$\begin{aligned}
& - \left(\frac{2}{\rho} \right)^{d/2} \lambda_d(\{M_\theta^{1/2} \widehat{W}_s : 0 \leq s \leq \tau\} \cap B(0, K')) \Bigg] \\
& = 0.
\end{aligned}$$

Since, by choosing K large enough, $P(\mathcal{V}^{[p]} \subset B(0, p^{1/2}K))$ can be made arbitrarily close to 1, uniformly in p , we have proved that

$$\begin{aligned}
(32) \quad p^{-d/2} \# \mathcal{V}^{[p]} & \xrightarrow[p \rightarrow \infty]{(P)} \left(\frac{2}{\rho} \right)^{d/2} \lambda_d(\{M_\theta^{1/2} \widehat{W}_s : 0 \leq s \leq \tau\}) \\
& = \left(\frac{2\sigma^2}{\rho} \right)^{d/2} \lambda_d(\{\widehat{W}_s : 0 \leq s \leq \tau\}).
\end{aligned}$$

The relations between the Brownian snake and super-Brownian motion [11], Theorem IV.4, show that the quantity $\lambda_d(\{\widehat{W}_s : 0 \leq s \leq \tau\})$ is the Lebesgue measure of the range of a super-Brownian motion (with branching mechanism $2u^2$) started from $(2/\rho)\delta_0$. Finally, simple scaling arguments show that the limit can be expressed in the form given in the theorem. \square

5. Open problems and questions.

5.1. *The probability of visiting a distant point.* Theorem 7 gives the asymptotic behavior of the probability that a branching random walk starting with a single particle at the origin visits a distant point $a \in \mathbb{Z}^d$. It would be of interest to have a similar result in dimension $d \geq 4$, assuming that θ is centered and has sufficiently high moments. When $d \geq 5$, a simple calculation of the first and second moments of the number of visits of a (see, e.g., the remarks following Proposition 5 in [13]) gives the bounds

$$C_1 |a|^{2-d} \leq \Pi_{\mu, \theta}^*(a \in \mathbf{R}) \leq C_2 |a|^{2-d}$$

with positive constants C_1 and C_2 depending on d, μ and θ . When $d = 4$, one expects that

$$\Pi_{\mu, \theta}^*(a \in \mathbf{R}) \approx \frac{C}{|a|^2 \log |a|}.$$

Calculations of moments give $\Pi_{\mu, \theta}^*(a \in \mathbf{R}) \geq c_1 (|a|^2 \log |a|)^{-1}$, but proving the reverse bound $\Pi_{\mu, \theta}^*(a \in \mathbf{R}) \leq c_2 (|a|^2 \log |a|)^{-1}$ with some constant c_2 seems a nontrivial problem. This problem, in the particular case of the geometric offspring distribution, and some related questions are discussed in Section 3.2 of [2].

5.2. *The range in dimension four.* With our previous notation R_n for the range of a random walk indexed by a random tree distributed according to $\Pi_\mu(\cdot | \#\mathcal{T} = n)$, Theorem 14 in [13] states that in dimension $d = 4$,

$$\frac{\log n}{n} R_n \xrightarrow[n \rightarrow \infty]{L^2} 8\pi^2 \sigma^4,$$

provided μ is the geometric distribution with parameter $1/2$, and θ is symmetric and has exponential moments. It would be of interest to extend this result to more general offspring distributions. It seems difficult to adapt the methods of [13] to a more general case, so new arguments would be needed. In particular, finding the exact asymptotics of $\Pi_{\mu, \theta}^*(a \in \mathbf{R})$ (see the previous subsection) in dimension $d = 4$ would certainly be helpful.

5.3. *Branching random walk with a general initial configuration.* One may ask whether a result such as Theorem 8 remains valid for more general initial configurations of the branching particle system: Compare with Propositions 20 and 21 in [13], which deal with the case $d \geq 4$ and require no assumption on the initial configurations. In the present setting, Theorem 8 remains valid, for instance, if we assume that the initial positions of the particles stay within a bounded set independently of p . On the other hand, one might consider the case where we only assume that the image of $p^{-1}X_0^{[p]}$ under the mapping $a \mapsto p^{-1/2}a$ converges weakly to a finite measure ξ on \mathbb{R}^d . This condition ensures the convergence of the (rescaled) measure-valued processes $X^{[p]}$ to a super-Brownian motion Y with initial value $Y_0 = \xi$, and it is natural to expect that we have, with a suitable constant C ,

$$(33) \quad p^{-d/2} \#\mathcal{V}^{[p]} \xrightarrow[p \rightarrow \infty]{(d)} C\lambda_d \left(\bigcup_{t \geq 0} \text{supp } Y_t \right).$$

For trivial reasons, (33) will not hold in dimension $d = 1$. Indeed, for $\frac{1}{2} < \alpha < 1$, we may let the initial configuration consist of $p - \lfloor p^\alpha \rfloor$ particles uniformly spread over $\{1, 2, \dots, \sqrt{p}\}$ and $\lfloor p^\alpha \rfloor$ other particles located at distinct points outside $\{1, 2, \dots, \sqrt{p}\}$. Then the preceding assumptions hold (ξ is the Lebesgue measure on $[0, 1]$), but (33) obviously fails since $\#\mathcal{V}^{[p]} \geq \lfloor p^\alpha \rfloor$. In dimension 2, (33) fails again, for more subtle reasons: One can construct examples where the descendants of certain initial particles that play no role in the convergence of the initial configurations contribute to the asymptotics of $\#\mathcal{V}^{[p]}$ in a significant manner. Still, it seems likely that some version of (33) holds under more stringent conditions on the initial configurations [in dimension 3 at least, the union in the right-hand side of (33) should exclude $t = 0$, as can be seen from simple examples].

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